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Over-Relaxation Applied to the MacCormack Finite-Difference Scheme

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An over-relaxation procedure is applied to the MacCormack finite-difference scheme in order to reduce the computation time required to obtain a steady-state solution. The implementation of this acceleration procedure to an existing computer program using the regular MacCormack method is extremely simple and does not require additional storage. The over-relaxation procedure does not alter the steady-state solution, which is second-order accurate. The method is first applied to Burgers' equation. A stability condition and an expression for the increase in the rate of convergence are derived. The method is then applied to the calculation of the hypersonic viscous flow over a flat plate, using the complete Navier–Stokes equations, and the inviscid flow over a wedge. Reductions in computing time by factors of 3 and 1.5, respectively, are obtained by over-relaxation.

INTRODUCTION

Because of the mixed elliptic–hyperbolic nature of the steady, compressible, Navier–Stokes equations, difficulties are encountered when attempting to solve these partial differential equations using a finite-difference technique. This problem is usually overcome by integrating the unsteady equations, which are mixed hyperbolic–parabolic in nature, forward in time until a steady-state solution is obtained. One of the most popular finite-difference schemes for solving fluid flow problems in this manner is MacCormack's scheme [1–2]. This explicit method is simple to program and is second-order accurate in time and space. However, this method often requires a large amount of computational time to obtain the steady-state solution because of the stability constraint. This is particularly true when a very fine mesh is employed.

To reduce computing time, some authors have developed implicit methods which are not limited by stability constraints, but are limited by accuracy considerations. However, the logic of these schemes is more complicated and the computational time required per step of integration is greater. In some cases, an implicit method may require more computational time to reach steady state than an explicit method. For example, Rudy *et al.* [3] have tested several numerical methods which solve a free shear layer problem using the compressible Navier–Stokes equations. They found that a sequential alternating-direction implicit (ADI) finite-difference procedure requires longer computing times to reach steady state than an explicit hopscotch

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finite-difference procedure. This is in spite of the fact that the ADI method permits a time step which is almost 10 times larger than the explicit scheme.

The goal of the present study has been to increase the rate of convergence of MacCormack's explicit finite-difference scheme by applying an over-relaxation technique. This method has first been applied to the one-dimensional Burgers' equation and has then been tested on two example flow problems: (1) the hypersonic viscous flow over the sharp leading edge of a flat plate; (2) the inviscid flow over a wedge.

Application to Burgers' Equation

For simplicity, the over-relaxation scheme is initially applied to Burgers' equation. This equation serves as a model equation for the Navier–Stokes equations.

Method

The one-dimensional Burgers' equation is

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2} - \sigma \frac{\partial u}{\partial x}\right) = 0, \tag{1}$$

where t and x represent time and distance in the x direction, respectively, $u(x, t)$ is a scalar function, and σ is a specified constant. After MacCormack's scheme is applied to the linearized form of Eq. (1),

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(cu - \sigma \frac{\partial u}{\partial x}\right) = 0 \tag{2}$$

(where c is a constant), over-relaxation can be applied to both sequences of predicted and corrected values in the following manner:

Predictor step:

$$\begin{aligned} \bar{v}_j^{n+1} &= u_j^n - \Delta t\, \frac{[cu_{j+1}^n - \sigma[(u_{j+1}^n - u_j^n)/\Delta x]] - [cu_j^n - \sigma[(u_j^n - u_{j-1}^n)/\Delta x]]}{\Delta x} \\ &= u_j^n - \nu(u_{j+1}^n - u_j^n) + \mu(u_{j+1}^n + u_{j-1}^n - 2u_j^n), \end{aligned} \tag{3}$$

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \bar{\omega}(\bar{v}_j^{n+1} - \bar{u}_j^n). \tag{4}$$

Corrector step:

$$\begin{aligned} v_j^{n+1} &= \bar{u}_j^{n+1} - \Delta t\, \frac{[c\bar{u}_j^{n+1} - \sigma[(\bar{u}_{j+1}^{n+1} - \bar{u}_j^{n+1})/\Delta x]] - [c\bar{u}_{j-1}^{n+1} - \sigma[(\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1})/\Delta x]]}{\Delta x} \\ &= \bar{u}_j^{n+1} - \nu(\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}) + \mu(\bar{u}_{j+1}^{n+1} + \bar{u}_{j-1}^{n+1} - 2\bar{u}_j^{n+1}), \end{aligned} \tag{5}$$

$$u_j^{n+1} = u_j^n + \omega(v_j^{n+1} - u_j^n). \tag{6}$$

In these equations, the v's are intermediate quantities, the u's denote the final predictions, $\bar{\omega}$ and ω are the over-relaxation parameters, and ν and μ are given by

$$\begin{aligned} \nu &= c\,\Delta t/\Delta x, \\ \mu &= \sigma\,\Delta t/\Delta x^2, \end{aligned} \tag{7}$$

where Δt is the time step and Δx is the mesh spacing. The regular MacCormack scheme is obtained by setting $\bar{\omega} = 1$ and $\omega = \frac{1}{2}$.

Clearly, the v quantities do not need to be stored in arrays. Equations (4) and (6) require no additional storage if a simple overwriting procedure is employed. For example, in Eq. (4) $\bar{u}_j{}^n$ can be overwritten by $\bar{u}_j^{n+1}$. The implementation of the over-relaxation steps does not affect the calculation of differences, which is the largest part of the computational effort for most problems. Also note that, when $\bar{\omega} \neq 1$, the corrected values are coupled with the predicted values and these also need to be initialized. Upon defining

$$V_j{}^n = \begin{bmatrix} \bar{u}_j{}^n \\ u_j{}^n \end{bmatrix} \tag{8}$$

and eliminating the v quantities in Eqs. (3)–(6), the following (vector) finite-difference equation is obtained:

$$V_j^{n+1} = A_0 V_j{}^n + A_1 V_{j+1}^n + A_{-1} V_{j-1}^n + A_2 V_{j+2}^n + A_{-2} V_{j-2}^n, \tag{9}$$

where the A_i's are the matrices

$$\begin{aligned}
A_0 &= \begin{bmatrix} 1 - \bar{\omega} & \bar{\omega}(1 + \nu - 2\mu) \\ \omega(1 - \bar{\omega})(1 - \nu - 2\mu) & 1 + \omega[\bar{\omega}(6\mu^2 - 4\mu + 1 - 2\nu^2) - 1] \end{bmatrix}, \\
A_1 &= \begin{bmatrix} 0 & \bar{\omega}(\mu - \nu) \\ \omega(1 - \bar{\omega})\,\mu & \omega\bar{\omega}[(2\mu - \nu)(1 - 2\mu) + \nu^2] \end{bmatrix}, \\
A_{-1} &= \begin{bmatrix} 0 & \bar{\omega}\mu \\ \omega(1 - \bar{\omega})(\mu + \nu) & \omega\bar{\omega}[(2\mu + \nu)(1 - 2\mu) + \nu^2] \end{bmatrix}, \\
A_2 &= \begin{bmatrix} 0 & 0 \\ 0 & \bar{\omega}\omega\mu(\mu - \nu) \end{bmatrix}, \\
A_{-2} &= \begin{bmatrix} 0 & 0 \\ 0 & \bar{\omega}\omega\mu(\mu + \nu) \end{bmatrix}.
\end{aligned} \tag{10}$$

Modified Equation

When over-relaxation is only applied to the corrected values ($\bar{\omega} = 1$), any function $u(x, t)$, defined by

$$u(j\,\Delta x, n\Omega\,\Delta t) = u_j{}^n, \tag{11}$$

where $\Omega = 2\omega$, satisfies the modified equation [4]:

$$\begin{aligned}\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} - \sigma\frac{\partial^2 u}{\partial x^2} &= -\frac{c^2 \Delta t}{2}(\Omega - 1)\frac{\partial^2 u}{\partial x^2} \\ &+ c\left[-\frac{\Delta x^2}{6} + (\Omega - 1)\,\sigma\,\Delta t + \frac{\Omega(3 - 2\Omega)}{6}(c\,\Delta t)^2\right]\frac{\partial^3 u}{\partial x^3} \\ &+ \left[\sigma\frac{\Delta x^2 - 6(\Omega - 1)\,\sigma\,\Delta t}{12} + \frac{\Omega(2\Omega - 3)}{2}\sigma(c\,\Delta t)^2\right. \\ &\left. - \frac{(4\Omega - 1)\,\Delta x^2 + 3\Omega(2\Omega^2 - 4\Omega + 1)(c\,\Delta t)^2}{24}c^2\,\Delta t\right]\frac{\partial^4 u}{\partial x^4} + \cdots. \qquad (12)\end{aligned}$$

Thus, the over-relaxation scheme is first-order accurate in time, but remains second-order accurate in space. However, the accuracy of the intermediate solutions is of no concern in this study. The over-relaxation process does not alter the steady-state solution, which remains second-order accurate for any value of $\bar{\omega}$. This fact can be seen from Eqs. (4) and (6), which become at the steady state:

$$\bar{u}_j = \bar{v}_j, \qquad (13)$$

$$u_j = v_j = (u_j + v_j)/2. \qquad (14)$$

Equations (13) and (14) are independent of the over-relaxation parameters and thus define the same solution given by the regular MacCormack scheme.

Equation (12) has been derived assuming that the over-relaxation scheme is $\Omega = 2\omega$ times faster than the regular scheme when $\bar{\omega} = 1$. The verification, a posteriori, of the consistency of the scheme proves the validity of this assumption.

Stability Analysis

A Fourier stability analysis has been applied to Eq. (9). It has been found [4] that the method defined by $(\nu, \mu, \bar{\omega}, \omega)$ is stable if the roots λ_1, λ_2 of the equation

$$\lambda^2 - 2P\lambda + Q = 0, \qquad (15)$$

in which

$$P = 1 - [(\bar{\omega} + \omega)/2] + (\bar{\omega}\omega/2)R, \qquad (16)$$

$$Q = (\bar{\omega} - 1)(\omega - 1), \qquad (17)$$

where

$$R = [1 - 4\mu\sin^2(\theta/2)]^2 - 4\nu^2\sin^2(\theta/2) - 2\nu i\sin\theta[1 - 4\mu\sin^2(\theta/2)]$$

satisfy the condition:

$$\mathrm{Max}(|\lambda_1|, |\lambda_2|) \leqslant 1 \qquad (18)$$

for all the values of θ in the interval $[0, \pi]$.

Note that since $\bar{\omega}$ and ω appear symmetrically in the expressions of P and Q, the domain of stability for a given (ν, μ) is symmetric with respect to the line $\bar{\omega} = \omega$ in the $(\bar{\omega}, \omega)$ plane. Also, this domain is included in the region bounded by the arcs of the hyperbolas given by the equations $Q = 1$ and $Q = -1$, since the condition $|\lambda_1\lambda_2| \leqslant 1$ must be enforced. This gives the necessary condition of stability

$$|(\bar{\omega} - 1)(\omega - 1)| \leqslant 1. \tag{19}$$

For the heat equation, obtained by setting $c = 0$ in Eq. (2), no scheme operating with a value of μ greater than $\frac{1}{2}$ is stable. For values of μ in the interval $[\frac{1}{4}, \frac{1}{2}]$ the domain of stability is the square defined by the conditions $\bar{\omega} \leqslant 2$ and $\omega \leqslant 2$. For values of μ smaller than $\frac{1}{4}$, the method is stable if Eq. (19) is satisfied and $\{4 - 2(\bar{\omega} + \omega) + [1 + (1 - 4\mu)^2]\bar{\omega}\omega\} \geqslant 0$. For example, with $\bar{\omega} = 1$, the stability condition is $4\mu(1 - 2\mu)\omega \leqslant 1$.

For the complete linearized Burgers' equation ($c \neq 0$ and $\sigma \neq 0$), it has not been possible to express a necessary and sufficient stability condition in the form of an algebraic relation between the parameters ν, μ, $\bar{\omega}$, and ω. Instead the roots λ_1 and λ_2 of Eq. (15) have been calculated numerically to determine the stable methods. In general, the stability condition is more restrictive than the conditions $\bar{\omega} \leqslant 2$ and $\omega \leqslant 2$ unless a very small time step Δt is employed. In particular, for some neighborhood of the point $\bar{\omega} = \omega = 2$, the scheme is always unstable.

Rate of Convergence

An estimate of the reduction in computing time achieved by over-relaxation is derived in this section.

Since the over-relaxation scheme is first-order accurate in time, sequences $\bar{t}_n$ and t_n can be defined such that

$$\bar{u}_j^n = u_j(\bar{t}_n) + O(\Delta t^2), \tag{20}$$

$$u_j^n = u_j(t_n) + O(\Delta t^2), \tag{21}$$

where $u_j(t)$ denotes the exact value of u at the point (x_j, t). In the above equations, the truncation terms involving Δx which are assumed to have negligible influence on the rate of propagation of the numerical solution are omitted. If the time step used in the computation of step $n + 1$ is Δt_{n+1}, the increase in the rate of convergence, Ω_{n+1}, by the over-relaxation process can then be defined by

$$\Omega_{n+1} = (t_{n+1} - t_n)/\Delta t_{n+1}. \tag{22}$$

A limiting value for Ω_{n+1} can be obtained as follows. Using the definition of t_n (Eq. (21)) and Eq. (3), which defines a forward integration step over a time interval Δt_{n+1}, gives the expression

$$\bar{v}_j^{n+1} = u_j(t_n + \Delta t_{n+1}) + O(\Delta t^2). \tag{23}$$

Equation (4) is an extrapolation formula which gives

$$\begin{aligned} \bar{t}_{n+1} &= \bar{t}_n + \bar{\omega}[(t_n + \Delta t_{n+1}) - \bar{t}_n] \\ &= \bar{t}_n + \bar{\omega}(\Delta\tau_n + \Delta t_{n+1}), \end{aligned} \tag{24}$$

where $\Delta\tau_n = t_n - \bar{t}_n$.

Similarly for the corrector step:

$$v_j^{n+1} = u_j(\bar{t}_{n+1} + \Delta t_{n+1}) + O(\Delta t^2) \tag{25}$$

and

$$\begin{aligned} t_{n+1} &= t_n + \omega[(\bar{t}_{n+1} + \Delta t_{n+1}) - t_n] \\ &= t_n + \omega[(\bar{\omega} - 1)\,\Delta\tau_n + (\bar{\omega} + 1)\,\Delta t_{n+1}]. \end{aligned} \tag{26}$$

Combining (22), (24), and (26) gives

$$\Omega_{n+1} = \omega(\bar{\omega} - 1)\frac{\Delta\tau_n}{\Delta t_{n+1}} + \omega(\bar{\omega} + 1) \tag{27}$$

and

$$\Delta\tau_{n+1} = (\bar{\omega} - 1)(\omega - 1)\,\Delta\tau_n + [\omega(\bar{\omega} + 1) - \bar{\omega}]\,\Delta t_{n+1}. \tag{28}$$

If Δt_{n+1} converges to Δt and if a strict inequality is assumed in (19), the sequence $\Delta\tau_n$ converges to $\Delta\tau$ given by [4]:

$$\Delta\tau = \frac{\omega(\bar{\omega} + 1) - \bar{\omega}}{1 - (\bar{\omega} - 1)(\omega - 1)}\,\Delta t \tag{29}$$

so that Ω_{n+1} converges to Ω given by

$$\Omega = \frac{2\bar{\omega}\omega}{1 - (\bar{\omega} - 1)(\omega - 1)}. \tag{30}$$

Ω represents an estimate of the number of times the over-relaxation scheme is faster than the regular scheme, when both schemes operate with the same time step. In the particular case $\bar{\omega} = 1$, the value of Ω becomes 2ω, which was previously derived. Note that Ω is a symmetric function of $\bar{\omega}$ and ω. The contours of constant values of Ω are hyperbolas orthogonal to the axis of symmetry of the domain of stability for specified (ν, μ). This domain is included in the nonshaded region of Fig. 1, but is not, a priori, identical to it. A theoretical study has been conducted [4] to determine, for various values of $c\Delta x/\sigma$, the maximum Ω for which the method is stable. This study has shown that the over-relaxation procedure gives better improvement in the rate of convergence when fine meshes are employed. These cases, for which the convergence of the regular scheme is very slow, are, in fact, the cases of most interest.

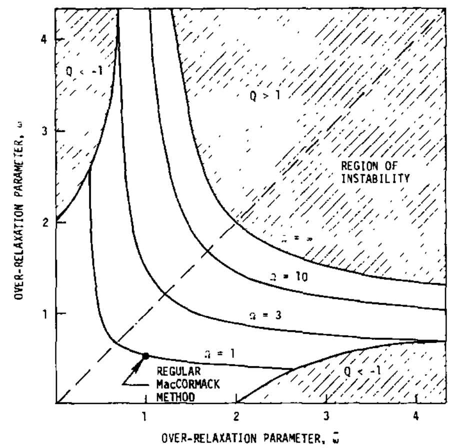


FIG. 1. A region including the domain of stability for specified ν and μ.

Numerical Tests

In the numerical tests involving Burgers' equation, the following fixed boundary conditions have been used:

$$u(0, t) = u_0, \qquad u(L, t) = 0 \qquad (t > 0). \tag{31}$$

The initial conditions are arbitrary and have been chosen to be

$$u(x, 0) = 0 \qquad (0 \leqslant x \leqslant L). \tag{32}$$

For the heat equation, the best convergence properties (without oscillations) have been obtained by setting $\bar{\omega} = \omega = 1.55$, which gives $\Omega = 6.9$. Figure 2 shows the convergence of the solutions obtained by application of the regular MacCormack scheme and the over-relaxation scheme. For the case presented, the over-relaxation process actually multiplies the rate of convergence by a factor of at least 10. Conver-

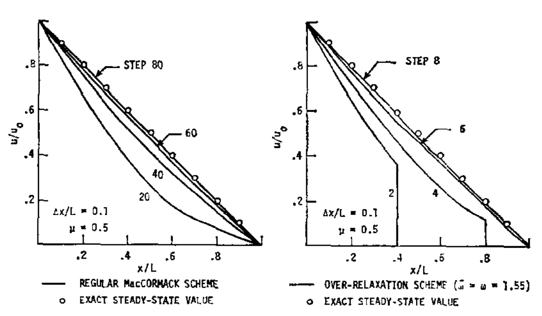


FIG. 2. Solutions to the heat equation obtained by application of the regular MacCormack scheme and the over-relaxation scheme.

gence can also be obtained with larger values of the over-relaxation parameters, provided that $\bar{\omega} < 2$ and $\omega < 2$. However, for these large values, the numerical solution converges to steady state in an oscillatory manner, which effectively decreases the rate of convergence.

The results for the linearized Burgers' equation are similar to those obtained for the heat equation. However, in this case, if $\bar{\omega}$ is set equal to ω, the optimum value of the over-relaxation parameters is very close to the limit of stability. For the case presented in Fig. 3, where $c\Delta x/\sigma = 0.5$, the maximum allowable time step for the regular scheme is given by $\mu = 0.533$. An actual increase in the rate of convergence by a factor of 6 is observed in the calculation while Eq. (30) predicts $\Omega = 5.3$.

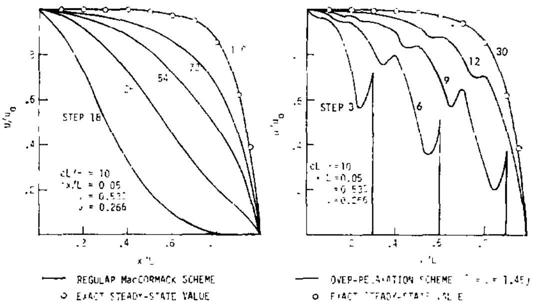


FIG. 3. Solutions to the linearized Burgers' equation ($cL/\sigma = 10$) obtained by application of the regular MacCormack scheme and the over-relaxation scheme.

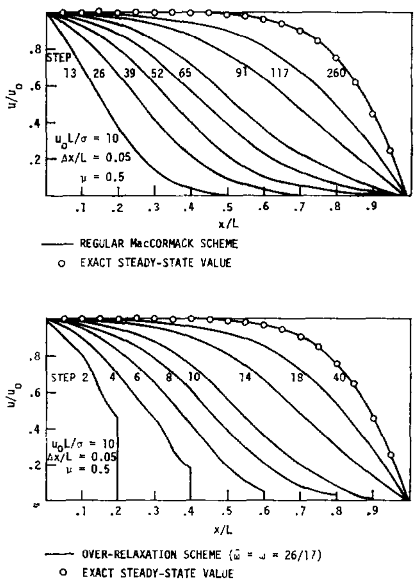


FIG. 4. Nonlinear Burgers' equation.

The nonlinear Burgers' equation with $u_0L/\sigma = 10$ and $\Delta x/L = 0.05$ has been integrated using the over-relaxation technique. For this case the experimental stability limit corresponds to $\mu = 0.52$, but the condition $\mu = 0.5$ has been used for simplicity. Using $\bar{\omega} = \omega = 26/17$ the rate of convergence is increased by a factor of 6.5, as shown in Fig. 4, and this agrees with the prediction of Eq. (30). It appears that Eq. (30) may underestimate the actual improvement obtained by applying over-relaxation. However, when the convergence of the numerical solution to steady state is slow (as in Fig. 4), this estimate is accurate because the passage to the limit $\Omega_{n+1} \to \Omega$ is justified.

In all cases, the steady-state solutions obtained by application of the regular scheme and the over-relaxation scheme are identical at all the grid points to at least six significant figures.

Navier–Stokes Solutions

The implementation of the over-relaxation scheme to an already existing computer program applying MacCormack's method is straightforward. Apart from an initialization procedure, it only requires the addition (or modification) of the two statements defined by Eqs. (4) and (6).

Hypersonic Viscous Flow over the Sharp Leading Edge of a Flat Plate

The present over-relaxation technique has been used to compute the hypersonic viscous flow over the sharp leading edge of a flat plate [5–6]. The flow conditions are those of [5, Case I], i.e.,

$$
\begin{aligned}
M_\infty &= 10.15 && \text{(freestream Mach number)},\\
\gamma &= 1.4 && \text{(ratio of specific heats)},\\
Pr &= 0.72 && \text{(Prandtl number)},\\
Re_\infty &= 4656/\text{ft} && \text{(freestream Reynolds number)},\\
p_\infty &= 0.0557\ \text{lb/ft}^2 && \text{(freestream pressure)},\\
T_\infty &= 250°\text{R} && \text{(freestream temperature)},\\
T_W &= 540°\text{R} && \text{(wall temperature)}.
\end{aligned}
$$

Figure 5 illustrates the computational domain which extends from the sharp leading edge to a point in the merged layer. This flow field cannot be calculated correctly using the boundary-layer equations or the thin-layer equations. At least the complete Navier–Stokes equations must be used. Even these are not valid in the rarefied region at the leading edge, but they appear to give reasonable results.

It was found that the regular MacCormack scheme required about 700 steps to converge to steady state when operating at the experimental stability limit. Utilizing the same stability limit, the scheme operating with $\bar{\omega} = 2\omega = 1.8$ has been found stable and convergent. This scheme was expected to be $\Omega = 3$ times faster, according

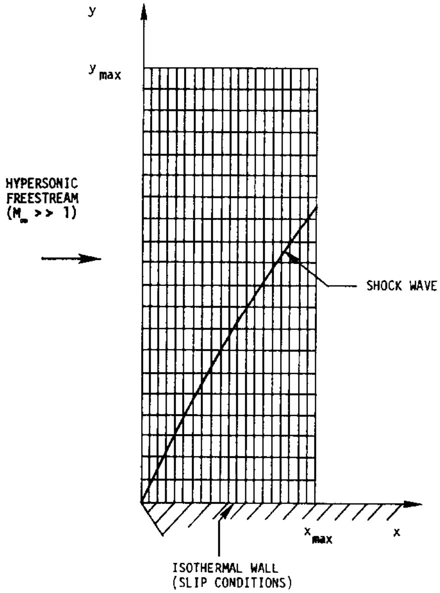


FIG. 5. Computational domain used in the calculation of the flow over a flat plate.

to Eq. (30), than the regular scheme and this was achieved in the computation. This is clearly demonstrated by Fig. 6 which shows the time history of the pressure distribution along the plate obtained by application of each scheme. The converged distributions (normal to the wall) of pressure at three different stations along the plate are shown in Fig. 7. Identical steady-state solutions are observed.

In conclusion, the over-relaxation procedure has reduced the computing time by a factor of 3 for this particular problem.

The Inviscid Flow over a Wedge

The inviscid flow over a wedge at supersonic speed (Fig. 8) has been computed using the present over-relaxation technique. The computer program is due to R. G. Hindman of Iowa State University. The case under study is defined by

$M_\infty = 2$ (freestream Mach number),

$p_\infty = 2117 \text{ lb/ft}^2$ (freestream pressure),

$\rho_\infty = 0.002377 \text{ slugs/ft}^3$ (freestream density),

$u_\infty = 2233 \text{ ft/sec}$ (freestream x component of velocity),

$v_\infty = 0$ (freestream y component of velocity),

$\theta_W = 5°$ (wedge half-angle).

This problem does not fall into the class of problems where the computing time to obtain the steady-state solution is large. This is due to the absence of viscous effects

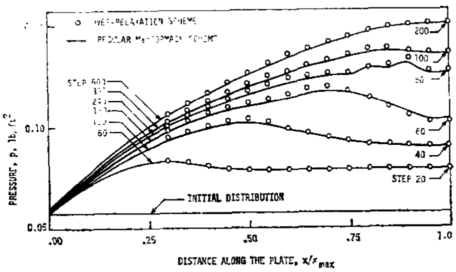


FIG. 6. Time histories of the pressure distribution along the plate obtained by applying the regular and the over-relaxation schemes ($\bar{\omega} = 2\omega = 1.8$).

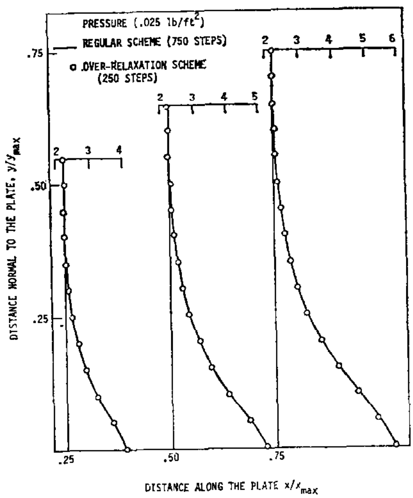
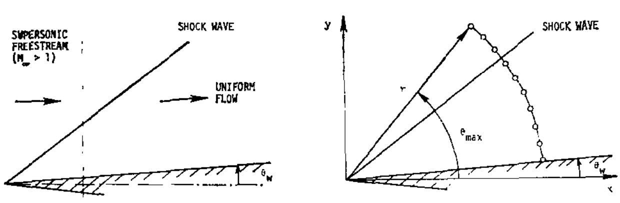


FIG. 7. Steady-state pressure distribution normal to the plate.



(a) (b)

FIG. 8. Flow pattern and computational mesh for wedge problem.

(no stress tensor to evaluate and no "viscous" stability limitation), and to the fact that only one spatial coordinate (the polar angle θ) needs to be retained because the flow is conical. The principal interest, here, has been to test the applicability of the over-relaxation technique to an inviscid flow problem with a sharp discontinuity (the shock wave).

This problem is governed by the Euler equations which are obtained by dropping the viscous terms in the Navier–Stokes equations. Polar coordinates have been employed with radial derivatives omitted for this conical flow.

When operating at the experimental stability limit, $\Delta t_{\max}$, the regular MacCormack scheme requires 440 steps to converge (to four significant figures). Successful results have been obtained by applying over-relaxation to the corrected values ($\bar{\omega} = 1$ and $\omega > \frac{1}{2}$). The fastest scheme operates with $\Delta t = 0.8\,\Delta t_{\max}$ and $\Omega = 2\omega = 1.90$. This scheme, which requires 280 steps to converge to four significant figures, is 1.57 times faster than the regular scheme. The converged pressure distributions obtained by application of both schemes are shown in Fig. 9. Slightly larger dispersive errors near the shock appear in the solution obtained by the over-relaxation scheme. These inaccuracies are not directly due to the application of over-relaxation but are due to the smaller time step which was used in order to maximize the rate of convergence. However, the shock is correctly located and the values for the pressure behind the shock obtained by both methods match. Figure 10 shows the solution

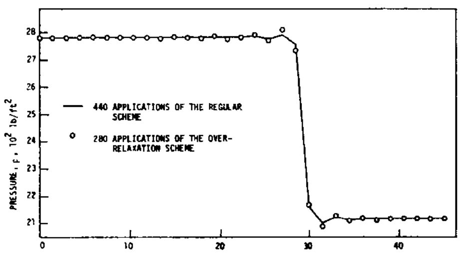


FIG. 9. Steady-state pressure distribution obtained by application of the regular scheme and the fastest over-relaxation scheme.

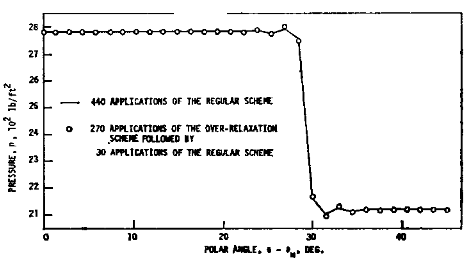


FIG. 10. Steady-state pressure distribution obtained when the regular method is used to terminate the iterative process.

obtained after 270 applications of the fastest over-relaxation scheme ($\Delta t = 0.8\ \Delta t_{\max}$) and 30 applications of the regular method ($\Delta t = \Delta t_{\max}$). This method produces the same value for the pressure behind the shock in 300 steps, which is 1.47 times faster than the regular scheme. This process reduces significantly the small difference between the two steady-state solutions in Fig. 9.

CONCLUSION

This study has shown that the rate of convergence to the steady-state solution using MacCormack's finite-difference scheme can be significantly increased by applying an over-relaxation procedure. The implementation of this acceleration procedure to an existing computer program is extremely simple. Apart from some initialization statements, it requires the modification of only two computer cards and no additional storage. In its general form, the scheme includes the regular method as a particular case.

With respect to the time-dependent equations of motion, the scheme becomes first-order accurate in time, but remains second-order accurate in space when over-relaxation is only applied to the sequence of corrected values. However, in the limit (as t tends to infinity), second-order accuracy is maintained. In particular, if an equal time step is employed, the over-relaxation scheme produces the same converged solution as the regular scheme but with fewer computations. The study of Burgers' equation has shown that the over-relaxation process gives better improvement in rate of convergence when fine meshes are employed. An example problem has been presented, where the process multiplies the rate of convergence by a factor of 6.5. Application of the over-relaxation scheme to the calculation of the hypersonic viscous flow over a flat plate and the inviscid flow over a wedge, resulted in reductions of computing time by factors of 3 and 1.5, respectively. It appears that the scheme is better suited to problems where the solution converges slowly and monotonically to steady state, which is typical of most viscous-dominated problems.

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